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Characterizations of Jacobi sign regular matrices ☆

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ABSTRACT

Several characterizations of Jacobi nonsingular sign regular matrices are presented. Moreover, a stable test of $\mathcal{O}(n)$ elementary operations is obtained to check if an $n \times n$ Jacobi nonsingular matrix is sign regular.

1. Introduction

Let $1 \leq k \leq \min\{m, n\}$ and fix a k-vector of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_j \in \{\pm 1\}$ for $j = 1, \ldots, k$, which is called a *signature*. An $m \times n$ matrix A is *sign regular of order k with signature ε* (SR_k) if, for each $j = 1, \ldots, k$, all minors of order j have the same sign ε_j or are zero. When $k = \min\{m, n\}$, a sign regular matrix of order k is simply called *sign regular* (SR) matrix. The interest of nonsingular SR matrices comes from their characterizations as variation-diminishing linear maps: the number of sign changes in the consecutive components of the image of a vector is bounded above by the number of sign changes in the consecutive components of the vector (cf. Theorems 5.3 and 5.6 of [1]). SR matrices have applications in many fields: approximation theory, computer aided geometric design, statistics, economics or numerical analysis (cf. [3,11,12]). If A is SR_k with signature ε with $\varepsilon_i = 1$ for all $i = 1, \ldots, k$, then we say that A is *totally positive of order k* (TP_k). When $k = \min\{m, n\}$, a totally positive matrix of order k is simply called *totally positive* (TP) matrix. Totally positive matrices form

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a very important subclass of SR matrices. Applications of TP matrices can be seen in [1,11] or [14]. Totally positive matrices are called by some authors totally nonnegative matrices. Other subclasses of SR matrices have been studied in [4,5,8,10] or [13].

A matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ is *Jacobi* (or tridiagonal) if $a_{ij} = 0$ whenever $|i - j| > 1$. SR Jacobi matrices play a very important role in computer aided geometric design, where they are matrices associated with corner cutting algorithms, the main family of algorithms in this field (cf. [12]). There are many known characterizations of TP matrices (see, for instance, [9] or [7]). The problem of obtaining characterizations of SR matrices is much more difficult than that of characterizing TP matrices. For instance, compare the test of $\mathcal{O}(n^3)$ elementary operations to check if an $n \times n$ nonsingular matrix is TP (see [9]) with the test of $\mathcal{O}(n^4)$ elementary operations to check if it satisfies strict sign regularity, a stronger property than sign regularity (see [6]). As recalled in Section 2, in the case of nonsingular Jacobi TP matrices, it is known that they are characterized by the positivity of their principal minors (cf. Proposition 2.4 and Corollary 3.8 of [1] or Theorem 4.3 of [14]). This paper presents characterizations of nonsingular Jacobi SR matrices.

Section 2 includes some basic notations and auxiliary results. In Section 3 we prove that, if A is a Jacobi nonsingular SR matrix, then either A or $-A$ is TP_{n-1}. In Section 4 we provide the main characterizations of this paper. In Theorem 4.1 (ii), they are characterized in terms of some submatrices. Several examples show that the conditions used in this characterization cannot be improved. Theorem 4.1 (iii) is used to provide a stable test of $\mathcal{O}(n)$ elementary operations to check if an $n \times n$ Jacobi nonsingular matrix is SR.

2. Basic notations and auxiliary results

Let us start by introducing basic notations. Given $k, l \in \{1, 2, \ldots, n\}$, let α (resp., β) be any increasing sequence of k (resp., l) positive integers less than or equal to n; their set is denoted by $Q_{k,n}$. For each $\alpha \in Q_{k,n}$, its dispersion number $d(\alpha)$ is defined by

$$d(\alpha) := \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i - 1) = \alpha_k - \alpha_1 - (k-1),$$

with the convention $d(\alpha) = 0$ for $\alpha \in Q_{1,n}$. Let us observe that in general $d(\alpha) = 0$ means that α consists of k consecutive integers. Let A be a real square matrix of order n. Then we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A containing rows numbered by α and columns numbered by β. Besides let $A[\alpha] := A[\alpha|\alpha]$.

Let us recall that SR matrices can be characterized by a reduced number of minors, as the following result shows (cf. Theorem 2.1 of [1]).

Theorem 2.1. *Let A be an $n \times m$ matrix with rank r. If*

$$\varepsilon_k \det A[\alpha \mid \beta] \geqslant 0, \quad \alpha \in Q_{k,n}, \ \beta \in Q_{k,m}, d(\beta) \leqslant m - r, k = 1, \ldots, \min(n, m), \tag{1}$$

then A is SR with signature ε.

Let us recall the Cauchy–Binet identity (cf. formula (1.23) of [1]) for the minors of the product of two matrices: if A and B are $n \times n$ matrices and $\alpha, \beta \in Q_{k,n}$, then

$$\det(AB)[\alpha|\beta] = \sum_{\gamma \in Q_{k,n}} \det A[\alpha|\gamma] \det B[\gamma|\beta].$$

Using the previous identity, one can derive the following slight generalization of Theorem 3.1 of [1].

Theorem 2.2. *If A, B are SR_r and SR_s matrices with signatures $\varepsilon(A)$, $\varepsilon(B)$ respectively, then the matrix AB is SR_t with $t = \min\{r, s\}$, and has signature $(\varepsilon_1(A)\varepsilon_1(B), \ldots, \varepsilon_t(A)\varepsilon_t(B))$.*

Let us now recall some elementary facts on Gaussian elimination, which transforms a linear system $Ax = b$ into an equivalent upper triangular linear system $Ux = c$. For nonsingular matrices A, Gaussian elimination (without pivoting) consists of a succession of at most $n - 1$ major steps resulting in a sequence of matrices as follows:

$$A = A^{(1)} \longrightarrow A^{(2)} \longrightarrow \cdots \longrightarrow A^{(n)} = U, \tag{2}$$

where $A^{(t)} = (a_{ij}^{(t)})_{1\le i,j\le n}$ has zeros below its main diagonal in the first $t-1$ columns. To obtain $A^{(t+1)}$ from $A^{(t)}$ we produce zeros in column t below the *pivot element* $a_{tt}^{(t)}$ by subtracting multiples of row t from the rows beneath it. It is well known that, if the leading principal minors $\det A[1, \dots, k]$ are nonzero for all $k = 1, \dots, n$, then Gaussian elimination can be preformed without row exchanges. Then, using the well known fact that the elements obtained under the elimination process can be obtained as Schur complements of submatrices, one has, for $t > 1$, that the pivot elements satisfy

$$a_{tt}^{(t)} = \frac{\det A^{(t)}[1, \dots, t]}{\det A^{(t-1)}[1, 2, \dots, t-1]} \tag{3}$$

The principal minors of a nonsingular TP matrix are positive (cf. Corollary 3.8 of [1]):

Proposition 2.3. *Let A be an $n \times n$ totally positive matrix. If A is nonsingular, then $\det A[\alpha] > 0$ for all k and $\alpha \in Q_{k,n}$.*

For a nonnegative Jacobi matrix, the nonnegativity of all its principal minors is a sufficient (and necessary) condition for its total positivity, as shown in Theorem 2.3 of [1]:

Proposition 2.4. *Let A be a Jacobi matrix of order n. If A is nonnegative and $\det A[\alpha] \geqslant 0$ when $d(\alpha) = 0$, then A is totally positive.*

It is known (cf. p. 100 of [14]) that if A is nonnegative and $\det A[1, \dots, k] > 0$, for all $k = 1, \dots, n$, then A is TP. The following result can be deduced from this one and provides a weaker sufficient condition for the total positivity of a nonnegative Jacobi matrix, which will be used later. For the proof, when $\det A = 0$, replace for each $\delta > 0$ the matrix A by the matrix A_δ with the entries of A except the (n, n) entry that is $a_{nn} + \delta$. Then $\det A_\delta > 0$ and so A_δ is TP for all $\delta > 0$. Thus, $A = \lim_{\delta\to 0} A_\delta$ and, since the set of TP matrices is closed, A is TP.

Lemma 2.5. *Let A be a Jacobi nonnegative $n \times n$ matrix ($n \geqslant 2$). If $\det A[1, \dots, k] > 0$ for $k = 1, \dots, n-1$ and $\det A \geqslant 0$ then A is TP.*

3. Jacobi SR matrices and TP_{n-1}-matrices

In this section we shall prove that a Jacobi, SR, nonnegative and nonsingular matrix must be TP_{n-1}. Let us see first that the hypothesis on nonsingularity is necessary. In fact, the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is Jacobi, nonnegative, SR and has a 2×2 submatrix with negative determinant: $\det A[1, 2] < 0$.

Let us observe that if the matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$ is Jacobi, SR, nonnegative and satisfies that $a_{i+1,i} \neq 0$ for all $i = 1, \dots, n-1$, then it is straightforward to check (by taking submatrices with those entries as diagonal entries) that A is TP_{n-1}. Analogously, the same conclusion holds if we require

$a_{i,i+1} \neq 0$ for all $i = 1, \ldots, n-1$. In order to deal with the general case, we first present two auxiliary results.

Lemma 3.1. *Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ $(n \geqslant 3)$ be a Jacobi, SR and nonsingular matrix. Then $a_{11} \neq 0$.*

Proof. If we assume that $a_{11} = 0$, then $a_{12} \neq 0$ and $a_{21} \neq 0$ because A is nonsingular and Jacobi. Then $\det A[1, 2] = -a_{12}a_{21} < 0$. But, since at least one of both $a_{n,n-1}$ or a_{nn} is nonzero, then $\det A[2, n \mid 1, j] > 0$ for $j = n-1$ or $j = n$, respectively, contradicting that A is SR. $\square$

Lemma 3.2. *Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ $(n \geqslant 3)$ be a Jacobi, SR, nonsingular and nonnegative matrix. Then we can perform a step of Gaussian elimination (see (2)) and the matrix $A^{(2)}[2, \ldots, n]$ is SR and nonnegative.*

Proof. By Lemma 3.1, $a_{11} \neq 0$ and so we can perform a step of Gaussian elimination to obtain $A^{(2)}$ (see (2)). If $a_{21} = 0$, then $A^{(2)} = A$ and the result is obvious. If $a_{21} \neq 0$, then one clearly has for any $k \leq n-1$ and indices $(2 \leq) i_1 \leq \cdots \leq i_k (\leq n)$ and $(2 \leq) j_1 \leq \cdots \leq j_k (\leq n)$ that

$$\det A^{(2)}[i_1, \ldots, i_k | j_1 \ldots, j_k] = \frac{\det A[1, i_1, \ldots, i_k | 1, j_1 \ldots, j_k]}{a_{11}} \tag{4}$$

and, since A is SR, we conclude that $A^{(2)}[2, \ldots, n]$ is also SR.

On the other hand, since A is Jacobi, $a_{31} = 0$, and, since A is nonsingular (and nonnegative), there exists $j > 1$ such that $a_{3j} > 0$. Then $\det A[1, 3|1, j] = a_{11}a_{3j} > 0$ and so all 2×2 submatrices of A have nonnegative determinant. Therefore, we deduce from (4) and $a_{11} > 0$ that for any $i, j > 1$, $a_{ij}^{(2)} \geq 0$, and the result follows. $\square$

We now give a first characterization of nonnegative Jacobi SR matrices.

Theorem 3.3. *Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ be a nonsingular nonnegative Jacobi matrix $(n \geqslant 2)$. Then A is SR if and only if A is TP_{n-1}.*

Proof. If A is TP_{n-1}, obviously A is SR because A is an $n \times n$ matrix.

Let us now assume that A is SR and we shall prove that A is TP_{n-1} by induction on n. If $n = 2$, it is obvious since A is nonnegative, we have that A is TP_{n-1}. If $n > 2$, let us assume that it holds for $n-1$ and let us prove it for n.

By Lemma 3.2, we can perform a step of Gaussian elimination and the matrix $A^{(2)}[2, \ldots, n]$ is SR and nonnegative. Since it is also a Jacobi matrix, by the induction hypothesis we have that $A^{(2)}[2, \ldots, n]$ is TP_{n-2}. Clearly, we can conclude that $A^{(2)}$ is TP_{n-2}.

Now, we have that

$$A = L_1 A^{(2)}, \qquad L_1 := \begin{pmatrix} 1 & & & \\ \frac{a_{21}}{a_{11}} & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Observe that the bidiagonal nonnegative matrix L_1 is TP. Since $A^{(2)}$ is TP_{n-2}, we deduce from Theorem 2.2 that A is also TP_{n-2}.

It remains to find a positive minor of order $n-1$ in order to prove that A is TP_{n-1}. Let us observe that, by (4),

$$\det A[1, \ldots, n-1] = a_{11} \det A^{(2)}[2, \ldots, n-1]. \tag{5}$$

Since $A^{(2)}[2, \ldots, n]$ is TP_{n-2},

$$\det A^{(2)}[2, \ldots, n-1] \geqslant 0 \tag{6}$$

Since $a_{11} > 0$ by Lemma 3.1, we have two possibilities:

- If $\det A^{(2)}[2, \ldots, n-1] > 0$ then, by (5), we have that $\det A[1, \ldots, n-1] > 0$ and we finish the proof.
- If $\det A^{(2)}[2, \ldots, n-1] = 0$, we can write $\det A^{(2)}[2, \ldots, n]$ as

$$a_{nn} \det A^{(2)}[2, \ldots, n-1] - a_{n-1,n}a_{n,n-1} \det A^{(2)}[2, \ldots, n-2]$$
$$= -a_{n-1,n}a_{n,n-1} \det A^{(2)}[2, \ldots, n-2].$$

Observe that $\det A^{(2)}[2, \ldots, n-2] \geqslant 0$ because $A^{(2)}$ is TP_{n-2} and this minor cannot be null because $A^{(2)}$ (and so $A^{(2)}[2, \ldots, n]$) is nonsingular. Thus, it is positive and so

$$\det A^{(2)}[2, \ldots, n] < 0.$$

Then $\det A = a_{11} \det A^{(2)}[2, \ldots, n] < 0$, and so we have that

$$\det A = a_{nn} \det A[1, \ldots, n-1] - a_{n-1,n} \det A[1, \ldots, n-2, n \mid 1, \ldots, n-1] < 0. \tag{7}$$

Since $\det A[1, \ldots, n-1] = 0$, then we deduce, by (7), that

$$\det[1, \ldots, n-2, n \mid 1, \ldots, n-1] > 0$$

and this is a positive minor of order $n-1$. □

Since an SR matrix is either nonpositive or nonnegative, we can deduce the following consequence from the previous result.

Corollary 3.4. *Let A be an $n \times n$ ($n \geqslant 2$) Jacobi and nonsingular matrix. Then A is SR if and only if A is TP_{n-1} or $-A$ is TP_{n-1}.*

In conclusion, Jacobi nonsingular SR matrices can present only the following 4 signature sequences: $(1, \ldots, 1)$, $(1, \ldots, 1, -1)$, $(-1, 1, \ldots, (-1)^n)$, or finally the signature $(-1, 1, \ldots, (-1)^{n-1}, (-1)^{n-1})$. Other results for SR matrices with signature $(1, \ldots, 1, -1)$ appear in [10].

The following result, consequence of Theorem 3.3, satisfies all hypotheses of Lemma 2.5, but replacing $\det A[1, \ldots, n-1] > 0$ by the condition $\det A[1, \ldots, n-1] = 0$ and adding the hypotheses of nonsingularity and sign regularity. The conclusion is different to that of Lemma 2.5.

Corollary 3.5. *Let A be a Jacobi nonsingular SR nonnegative $n \times n$ matrix ($n \geqslant 2$). If* $\det A[1, \ldots, k] > 0$ *for $k = 1, \ldots, n-2$ and* $\det A[1, \ldots, n-1] = 0$,*then A has signature* $(1, \ldots, 1, -1)$.

Proof. By Theorem 3.3, either A is TP or A has signature $(1, \ldots, 1, -1)$. The result follows from Proposition 2.3, since A cannot be TP because it is nonsingular and $\det A[1, \ldots, n-1] = 0$. □

4. Main characterizations

In this section we present the main result of the paper. We characterize nonsingular Jacobi SR matrices by some principal submatrices and also in an algorithmic way by means of Gaussian elimination.

Theorem 4.1. *Let A be an $n \times n$ $(n \geqslant 3)$ nonsingular nonnegative Jacobi matrix. Then the following conditions are equivalent:*

(i) *A is SR.*
(ii) *The submatrix $A[1, \ldots, n-1]$ is TP and $A[1, \ldots, n-2]$ is nonsingular and besides the submatrix $A[2, \ldots, n]$ is TP and $A[2, \ldots, n-1]$ is nonsingular.*
(iii) *If $p_1, \ldots, p_{n-1}$ are the first $n-1$ pivots of the Gaussian elimination of A and $q_1, \ldots, q_{n-1}$ are the pivots of the Gaussian elimination of $A[2, \ldots, n]$, then*

$$p_1, \ldots, p_{n-2} > 0, \quad q_1, \ldots, q_{n-2} > 0 \quad \text{and} \quad p_{n-1}, q_{n-1} \geqslant 0.$$

Proof. (i) ⇒ (ii). Since A is Jacobi SR, by Theorem 3.3 A is TP_{n-1}. So, the submatrices $A[1, \ldots, n-1]$ and $A[2, \ldots, n]$ are TP. Let us see now that $A[1, \ldots, n-2]$ and $A[2, \ldots, n-1]$ are nonsingular. If $\det A > 0$ then A is TP and nonsingular. So, its principal minors are positive (by Proposition 2.3) and thus, we have that $\det A[1, \ldots, n-2] > 0$ and $\det A[2, \ldots, n-1] > 0$. Let us assume that $\det A < 0$. Since

$$\det A = a_{nn} \det A[1, \ldots, n-1] - a_{n-1,n}a_{n,n-1} \det A[1, \ldots, n-2], \tag{8}$$

$a_{nn} \geqslant 0$ and $\det A[1, \ldots, n-1] \geqslant 0$, we can deduce that $\det A[1, \ldots, n-2] > 0$. Besides, since

$$0 < \det A[1, \ldots, n-2] = a_{11} \det A[2, \ldots, n-2] - a_{1,2}a_{2,1} \det A[3, \ldots, n-2],$$

we can say that $\det A[2, \ldots, n-2] > 0$. And since $0 \leqslant \det A[2, \ldots, n] = a_{nn} \det A[2, \ldots, n-1] - a_{n-1,n}a_{n,n-1} \det A[2, \ldots, n-2]$, we conclude that $\det A[2, \ldots, n-1] > 0$.
(ii) ⇒ (i). Since A is nonsingular we know, by Theorem 2.1, that A will be SR if $\varepsilon_k \det A[\alpha \mid \beta] \geqslant 0$, where $k = 1, \ldots, n$ and $\alpha, \beta \in Q_{k,n}$, $d(\beta) = 0$. Since A is square, it is sufficient to see it for all $k \leqslant n-1$.

If $\alpha, \beta \subseteq \{1, \ldots, n-1\}$, then $\det A[\alpha \mid \beta] \geqslant 0$ because of the total positivity of $A[1, \ldots, n-1]$.

If $\alpha_k = n \neq \beta_k$, then $\det A[\alpha \mid \beta] = a_{n,\beta_k} \det A[\alpha_1, \ldots, \alpha_{k-1} \mid \beta_1, \ldots, \beta_{k-1}]$ (observe that $a_{nj} = 0$ for all $j \in \{\beta_1, \ldots, \beta_{k-1}\}$). Besides, we have that $\det A[\alpha_1, \ldots, \alpha_{k-1} \mid \beta_1, \ldots, \beta_{k-1}] \geqslant 0$, because $A[1, \ldots, n-1]$ is TP and the sets $\{\alpha_1, \ldots, \alpha_{k-1}\} \subseteq \{1, \ldots, n-1\}$ and $\{\beta_1, \ldots, \beta_{k-1}\} \subseteq \{1, \ldots, n-2\}$. Then $\det A[\alpha \mid \beta] \geqslant 0$.

If $\alpha_k \neq n = \beta_k$, this case is analogous to the previous case, so we have $\det A[\alpha \mid \beta] = a_{\alpha_k,n} \det A[\alpha_1, \ldots, \alpha_{k-1} \mid \beta_1, \ldots, \beta_{k-1}] \geqslant 0$.

If $\alpha_k = \beta_k = n$, let us notice that $\beta_1 > 1$, because $d(\beta) = 0$ and $\beta \in Q_{k,n}$, with $k \leqslant n-1$. Thus we have two subcases:

If $\alpha_1 > 1$, then $\det A[\alpha \mid \beta] \geqslant 0$ because of the total positivity of $A[2, \ldots, n]$. Finally, if $\alpha_1 = 1$ then the case is similar to the case $\alpha_k = n \neq \beta_k$ considered above.
(ii) ⇒ (iii). Since $A[1, \ldots, n-2]$ is TP and nonsingular, all its principal minors are positive (by Proposition 2.3). Then, by (3), all its pivots $p_1, \ldots, p_{n-2}$ are positive. Analogously, the pivots $q_1, \ldots, q_{n-2}$ are also positive. Since $0 \leqslant \det A[1, \ldots, n-1]$ and $0 < \det A[1, \ldots, n-2]$, we deduce again from (3) that $p_{n-1} \geqslant 0$. Analogously, $q_{n-1} \geqslant 0$.
(iii) ⇒ (ii). The positivity of the pivots $p_1, \ldots, p_{n-2}$ implies by (3) that $\det A[1, \ldots, k] > 0$ for $1 \leq k \leq n-2$ and using that $p_{n-1} \geq 0$ we deduce from (3) that $\det A[1, \ldots, n-1] \geq 0$. Then, by Lemma 2.5, we conclude that the matrix $A[1, \ldots, n-1]$ is TP. Analogously, the positivity of the pivots $q_1, \ldots, q_{n-2}$ implies that the matrix $A[2, \ldots, n]$ is TP. The submatrices $A[1, \ldots, n-2]$ and $A[2, \ldots, n-1]$ are obviously nonsingular because we have seen that they have positive determinant. □

Taking into account Proposition 2.3 and Lemma 2.5, $n \times n$ nonsingular Jacobi TP matrices can be characterized in terms of the leading principal minors $\det A[1, \ldots, k]$, $k = 1, \ldots, n$. However, Theorem 4.1 (iii) characterizes $n \times n$ nonsingular Jacobi SR matrices using also the submatrix $A[2, \ldots, n]$.

The following matrix A shows the necessity of this condition:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 4 \\ 0 & 2 & 1 \end{pmatrix}.$$

It satisfies $\det A[1] = 1 > 0$, $\det A[1, 2] = 1 > 0$ and $\det A = -7 < 0$. But it is not SR because $\det A[2, 3] = -5 < 0$ and $\det A[1, 2] > 0$. Besides, conditions (ii) cannot be replaced by $A[1, \ldots, n-1]$ and $A[2, \ldots, n]$ are TP and nonsingular. In other words, requiring that $A[1, \ldots, n-1]$ and $A[2, \ldots, n]$ are nonsingular TP is not a necessary condition for the sign regularity of A. In fact, the SR matrices

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

satisfy $\det B[1, 2] = 0$ and $\det C[2, 3] = 0$.

Let us observe that the algorithmic test suggested by (iii) of the previous theorem can be applied to any nonsingular Jacobi $n \times n$ SR matrix because, if A is nonpositive, then we apply it to $-A$. The test stops when one of the conditions (iii) does not hold, and it takes into account Theorem 4.1, Lemma 2.5, Corollary 3.4 and Corollary 3.5.

Test to check if a Jacobi nonsingular matrix $A \geq 0$ is SR:

```
for i = 1 to n − 2
    p_i = ith pivot of A
    if p_i ⩽ 0 then
        return A is not SR.
    endif
endfor
p_{n−1} = (n − 1)th pivot of A
if p_{n−1} < 0 then
    return A is not SR.
else
    p_n = nth pivot
    if (p_{n−1} > 0) and (p_n ⩾ 0) then
        return A is TP.
    else
        for i = 1 to n − 2
            q_i = ith pivot of A[2, . . . , n − 1]
            if q_i ⩽ 0 then
                return A is not SR.
            endif
        endfor
        q_{n−1} = (n − 1)th pivot of A[2, . . . , n − 1]
        if q_{n−1} < 0 then
            return A is not SR.
        else
            return A is SR with signature (1, . . . , 1, −1).
        endif
    endif
endif
```

Let us see that this test has a computational cost of $\mathcal{O}(n)$ elementary operations which is very low. It performs Gaussian elimination to two Jacobi matrices of order n and $n-1$, respectively. Since the matrices are Jacobi, at each step of Gaussian elimination it is only needed to compute the pivot

$$p_i = a_{ii}^{(i-1)} - a_{i-1,i}^{(i-1)} \frac{a_{i,i-1}^{(i-1)}}{a_{i-1,i-1}^{(i-1)}} = a_{ii} - a_{i-1,i} \frac{a_{i,i-1}}{p_{i-1}},$$

which requires three elementary operations. Observe that $p_1 = a_{11}$ and the test also uses p_n. A similar cost corresponds to the computation of the pivots $q_1, \ldots, q_{n-1}$. So we have that the computational cost of the test is $2n-3$ subtractions, multiplications and divisions. Finally, taking into account [2], we conclude that checking with this test that A is SR is stable because we apply Gaussian elimination to two TP matrices.

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